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One-dimensional hydrogen atom: a singular potential in quantum mechanics

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Abstract. A generalized Laplace transform approach is developed to study the eigenvalue problem of the one-dimensional singular potential $V = -e^2/|x|$. Matching of solutions at the origin that has been a matter of much controversy is, thereby, made redundant. A discrete and non-degenerate bound-state spectrum results. Existing arguments in the literature that advocate (a) a continuous spectrum, (b) a degeneracy of energy levels as a result of a hidden $O(2)$ symmetry, (c) an infinite negative energy state and (d) an impenetrable barrier at the origin are found to be untenable. It is argued that a judicious use of generalized functions, coupled with some classical considerations, enables the conventional method of solving the problem to recover precisely the same results which are shown to be in accord with an accurate semiclassical analysis of the problem.

1. Introduction

Despite a long and cherished history of quantum mechanics, our understanding of singular potentials is less than adequate. Accepting that their relevance may be open to question, it would nevertheless be satisfying to be able to deal with singular potentials in the same manner as one is able to deal with non-singular potentials.

Rather than pursuing a purely formal approach, a more rewarding procedure might be to focus attention on specific but tractable models of such potentials. Here, we are concerned with the exactly solvable problem of the one-dimensional potential $V = -e^2/|x|$. Note that being a function of $|x|$ it is not analytic (e.g. $x = 0$ is not just a pole), and the independent variable spans the whole x-axis including the origin. Any solution should satisfy the wave equation over the entire range of permitted values of x.

The misleading formal resemblance of this problem to its three-dimensional counterpart has earned it the name 'one-dimensional hydrogen atom' problem, or briefly, the 1H atom problem. Historically, the 1H atom wave equation first appears as an approximation in the theory of the exciton in a strong magnetic field (Loudon 1959). From the intimate mathematical relationship between the Coulomb and oscillator systems (e.g. Quigg and Rosner 1979) it naturally suggests itself as a worthy exercise in Schrödinger theory. Being solvable it may serve to unravel some peculiarities of quantum mechanical behaviour in a singular potential field.

The exact solution to the problem first appears in the pioneering work of Loudon (1959). His detailed exposition reveals two unusual features of this one-dimensional system. First,

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its dimensionality notwithstanding, the discrete bound-state spectrum turns out to be doubly degenerate. Secondly, the normal state (nodeless) has infinite negative energy. A much more recent work by Davtyan *et al* (1987), that employs momentum representation, claims to confirm these findings and ascribes them to a hidden $O(2)$ symmetry. That a three-dimensional Coulomb problem has $O(4)$ symmetry and its two-dimensional counterpart has $O(3)$ symmetry makes this deduction seem quite natural and hence very persuasive. Unfortunately this result is incorrect as we show at the end of the paper. It is also significant that the presence of a hidden symmetry connection allows a purely algebraic treatment of the analogous Coulomb problems in higher dimensions whereas no such facility seems to be forthcoming in the one-dimensional case. However, the contribution of Davtyan *et al* (1987) is valuable for a plurality of reasons and we shall return to it later.

From the literature we have found that the aforementioned features have become a source of controversy and confusion. A comprehensive list of references appears in Bateman *et al* (1992) who assert that, since the potential is symmetric about $x = 0$, it suffices to consider the problem on the half-line whereas the singularity at $x = 0$ merits no special consideration. Benefiting from the Coulomb-oscillator duality mentioned above, they conclude that there are no even-parity solutions present and the features found by Loudon vanish. Arguable as their approach may be, their results seem to be correct. Unfortunately, they offer no explanation for how Loudon's conventional treatment and that of Davtyan *et al* (1987) ran into trouble.

Many authors have focused attention on the viability of even-parity solutions. An intuitive approach, relying on the general expectation that the zero angular momentum ($\ell = 0$) solutions of a three-dimensional problem correspond to the odd parity solutions of its one-dimensional counterpart, by Nieto (1979), suggests that an infinitely high barrier at $x = 0$ should somehow be present thereby excluding the even solutions. Andrews (1976) argues that, in spite of being attractive, the singularity at $x = 0$ is impenetrable, so that the problem effectively divides into two disjoint parts. A two-fold degeneracy is thus inevitable but no $O(2)$ symmetry is implied (Andrews 1988). We shall demonstrate that although one may rightfully choose to think of a barrier present at $x = 0$, any debate over its penetrability is neither essential nor enlightening. For quantum mechanics, that deals with wavefunctions, it is utterly irrelevant (cf Gostev and Frenkin 1987, 1988), whereas a classical particle with finite energy can traverse through the origin (see section 2 below).

Heines and Roberts (1969) contribute a further complication via a critical analysis of the solutions of the differential equation in hand. Besides confirming a discrete negative-energy spectrum corresponding to odd-parity solutions, they also discover a continuous spectrum that is unbounded from below. This inference is based crucially on how the solutions should be joined at the origin, precisely the point where the potential is singular. Accordingly, we also discuss this matter, remarking here by the way that Heines and Roberts failed to show that their wavefunctions corresponding to a continuous spectrum are mutually orthogonal. When dealing with singular potentials, it is indeed advisable to do so, as has been stressed by Case (1950). Moreover, Andrews (1988) has already correctly observed that, at least, the lowest group of their solutions do not form an orthogonal set (the situation is not rectifiable using the Schmidt orthogonalization procedure).

Another way to deal with the singularity at $x = 0$ is to smooth the potential (Loudon 1959, Heines and Roberts 1969, Mehta and Patil 1978, van Haeringen 1978, Gesztesy 1980). Strictly speaking, the 'point of singularity' is to be excluded from the domain of the Schrödinger operator which as a result becomes not self-adjoint and requires an extension. The 1H atom problem has been considered from that point of view by Gesztesy (1980) and by Gostev *et al* (1987), and their results justify in a sense our reasoning in section 4. Finally, the work of Gomes and Zimerman (1980) using the virial theorem is worth mentioning. It recovers only the odd-parity solutions and a discrete, non-degenerate spectrum, precisely the outcome of our study.

The primary purpose of this work is to examine the problem using integral transforms. However, in view of the background presented in the foregoing coupled with the fact that one-dimensional classical motion of a particle in the singular potential field has, to the best of our knowledge, never been discussed previously, we section the study as follows.

Section 2 discusses some interesting classical aspects of the problem. One classical feature will prove to be helpful in the analysis of the quantum motion of the particle. The effect of smoothing the potential on the classical motion is also commented upon.

A thorough semiclassical treatment of the motion in the corresponding singular potential well is the subject of section 3.

Section 4 considers the problem of matching the solutions at $x = 0$. The purpose of this section is to systematically trace the roots of various controversies that have been mentioned above. Eventually, we arrive at the correct solution to the problem using a conventional quantum mechanical approach and a proper matching relation.

The principal contribution of this work is reserved for section 5. Here, the solution to the problem is accomplished using integral transforms. It will be seen that this approach is ideally suited to deal with the problems posed by the singularity at $x = 0$. There is no need to directly address the problem of matching the solutions in this method.

Section 6 summarizes the results.

2. Classical motion and smoothing of the potential

Before embarking on the quantum mechanics of the given system, we consider certain classical aspects of the problem from which we hope to benefit in the sequel. Recall that whenever possible, a quantum Hamiltonian is obtained from its classical counterpart using the substitution rule.

Incidentally, as has been pointed out by Nieto (1979), the one-dimensional potential $-2\pi e|x|$ rather than $e/|x|$ corresponds to a positive charge at the origin which generates the force field

$$F'(x) = (-e)(-\mathrm{d}V/\mathrm{d}x) = -2\pi e^2\epsilon(x)$$

on an electron. Here $\epsilon(x)$ is a step-function sgn$x$ that equals $+1$ for positive x and -1 for negative x. In three-dimensions, this is equivalent to the field due to the coordinate plane YOZ with uniform surface charge density $\sigma = e$. Notice parenthetically that any physical 'one-dimensional' potential requires a source that is infinite in the OY and OZ directions, so that $V(x)$ must actually tend to infinity and not to zero as $|x| \to \infty$. Therefore the 1H atom problem rather corresponds to a one-dimensional motion in the Coulomb field $-e^2/r$ than a motion in true one-dimensional potential.

Comparing the previous expression for the force with the case in hand, we find (rigour notwithstanding)

$$F(x) = e^2\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{|x|}\right] = e^2\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\epsilon(x)}{x}\right] = -\frac{e^2\epsilon(x)}{x^2} + 2e^2\frac{\delta(x)}{x}. \tag{1}$$

Clearly, this force is not purely attractive as it contains a repulsive singularity contributed by the second term on the right-hand side of equation (1). Thus it appears that the potential $e/|x|$ disguises an 'infinitely high' (but 'infinitely narrow') repulsive barrier at the origin.

To visualize the motion in such a field we consider its smoothed version using a limiting representation for the step-function $\epsilon(x)$, say, for example,

$$\epsilon(x) = \lim_{\alpha\to 0} \tanh(x/\alpha).$$

Then:

$$F(x) = -\frac{e^2}{x^2}\left[\tanh\frac{x}{\alpha} - \frac{x}{\alpha}\left(\cosh\frac{x}{\alpha}\right)^{-2}\right]$$

which, for small x, becomes

$$F(x) \simeq -\frac{2e^2}{3\alpha^3}x.$$

This result is striking. The force provides harmonic oscillations about the point of stable equilibrium $x = 0$, whose frequency grows unchecked as $\alpha \to 0$. Thus, the second (repulsive) term in equation (1) makes the force to vanish (not infinitely increase!) at $x = 0$, but it does not forbid a 'penetration' of the electron through the origin, that is in contrast with the following confusing observation due to Andrews (1976): 'even an attractive singular potential might act as an impenetrable barrier in quantum mechanics'.

Incidentally, a truncated potential

$$V(x) = \begin{cases} e/|x| & |x| \geqslant b \\ e/b & |x| \leqslant b \end{cases}$$

with $b \to 0^+$, employed by Loudon (1959), behaves exactly like $2\delta(x)$ at $x = 0$, so that $F(x) = 0$ at $x = 0$ once again. However, another truncated potential $V(x) = e/(|x| + b)$, $b \to 0^+$, used by Loudon as well as by Heines and Roberts (1969), van Haeringen (1978) and Gesztesy (1980), does not satisfy the requirement $F(x) = 0$ at $x = 0$, and, in our view, it is not a suitable smoothed out version of the potential $e/|x|$ (to say nothing of the Coulomb-oscillator duality). Given its singular nature, smoothing of the potential is not an advisable operation, but our basic conclusions happen to be correct in the given context. This assertion is justified by the exact solution to the problem that we briefly review below.

For bounded motions we put the total energy E to be negative and set it equal to $-|E|$, so that,

$$-|E| = \frac{m}{2}\left(\frac{dx}{dt}\right)^2 - \frac{e^2}{|x|}$$

whence

$$\frac{dx}{dt} = \pm\left[\frac{2}{m}\left(\frac{e^2}{|x|} - |E|\right)\right]^{\frac{1}{2}}. \tag{2}$$

At the turning points, $v_x = dx/dt = 0$. Classical motion is thus possible in the range $|x| \leqslant A$, with the amplitude $A = e^2/|E|$. (We recall that the large semi-axis in the three-dimensional Kepler problem is $a = e^2/2|E| = A/2$). Focusing on $x > 0$, we routinely arrive at the result

$$|x| = a(1 - \cos\xi) \tag{3}$$

where ξ is a parameter related to time t by:

$$\omega t = \xi - \sin\xi \qquad \omega = \sqrt{e^2/ma^3}. \tag{4}$$

To find a solution valid for all x in the allowed range, we must match its separate parts at $x = 0$. For classical motion, it seems natural to require $x(t)$ to be continuous and the direction of motion (sign of v_x) to be retained at the point. With this proviso, it suffices to multiply equation (3) by a periodic step-function $\phi(\xi) = [\sin(\xi/2)]/|\sin(\xi/2)|$ of period 4π which is equal to $\epsilon(\xi)$ for $-2\pi \leqslant \xi \leqslant 2\pi$, in view of the fact that during half a period, according to equation (3), x increases from zero to $2a$ as ξ varies from zero to π, then it decreases back to zero as ξ continues from π to 2π, going on to negative values for the second half-period. Thus the complete solution is

$$x(t) = 2a[\sin(\xi/2)]^3/|\sin(\xi/2)| \tag{5}$$

which is seen to satisfy (as it should) the equation of motion by substitution in equation (2) (an extra δ-function, resulting from differentiation $\epsilon(\xi)$, is ineffective being multiplied by the function (3) which vanishes at $\xi = 0$).

The solution (5) has been found in a parametric form. Its explicit dependence on time is given by the Fourier expansion:

$$x(t) = \sum_{m=1}^{\infty} A_m \sin\frac{m\tau}{2} \qquad A_m = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} x[\xi(t)]\sin\frac{m\tau}{2}\,\mathrm{d}\tau$$

where $\tau = \omega t$, ω is given by (4). Substitution from (5) and integration by parts yield:

$$\begin{aligned} A_m &= \frac{a}{\pi}\int_0^{2\pi}(1-\cos\xi)\sin\frac{m\tau}{2}\,\mathrm{d}\tau \\ &= -\frac{2a}{\pi m}\int_0^{2\pi}\cos\frac{m\tau}{2}\,\mathrm{d}(\cos\xi) = \frac{2a}{\pi m}\int_0^{2\pi}\sin\xi\cos[m(\xi-\sin\xi)/2]\,\mathrm{d}\xi. \end{aligned}$$

Splitting the interval of integration into two equal parts: $(0, 2\pi) = (0, \pi) + (\pi, 2\pi)$, one can see that A_m are different from zero for odd $m = 2k + 1$ only, being proportional then to derivatives of the Weber's functions,

$$E_\nu(z) = \frac{1}{\pi}\int_0^{\pi}\sin(\nu\theta - z\sin\theta)\,\mathrm{d}\theta$$

see, for example, section 8.58 in Gradshteyn and Ryzhik (1965).

This enables one to express (5) as the following series:

$$x(t) = -2a\sum_{k=0}^{\infty} E'_{k+1/2}(k+1/2)\sin(k+1/2)\omega t]/(k+1/2) \tag{6}$$

which is to be compared with the Bessel solution of the Kepler problem for an elliptic orbit of eccentricity e:

$$x_e(t) = a\left[-3e/2 + 2\sum_{n=1}^{\infty} J'_n(ne)[\cos(n\omega t)]/n\right] \tag{7}$$

that can be found for example in Watson (1958). The noteworthy features of the above solution are as follows.

(i) The classical motion is described by an odd function $x(t)$ with only odd harmonics in the Fourier series representation of $x(t)$ surviving. (In the quantum and semiclassical cases, we shall find that only odd-parity wavefunctions are allowed.)

(ii) The amplitude of oscillations is twice as large as the semimajor axis of the corresponding elliptic orbit for the same energy. The period of motion is also twice that for Kepler motion. Such a doubling technically results from the fact that, as angular momentum assumes zero value, the leading singularity r^{-2} in the energy equation vanishes, leaving only the 'milder' term $|x|^{-1}$ in equation (2) which corresponds to another differential equation.

(iii) The function $x(t)$ is continuous and differentiable as a generalized function.

(iv) At $x = 0, v_x = \infty$, which is not surprising in view of the singular character of potential. It implies that the classical particle spends negligible time near the origin. (Physically, this seems to be the reason the particle escapes falling to the centre.) Quantum mechanically, it should mean that the particle is unlikely to be found near the origin, i.e. its wavefunction should vanish there. We must hasten to add that it is not incumbent upon a quantum particle to closely mimic the classical expectation in this respect, a classic example being provided by the case of a simple harmonic oscillator (the interested reader is referred to Shankar, 1981). The fact that $v_x \to \infty$ as $x \to 0$ also suggests that the classical problem be better treated relativistically, but our interest being in the quantum motion, such a digression would serve no additional purpose here.

(v) Quite important, motion with finite total energy is permitted, in spite of the divergence of the potential and kinetic energy contributions at $x = 0$.

(vi) The presence of a penetrable repulsive barrier in the Newtonian force stands vindicated by the exact solution. The 'amplitude' of velocity is infinite while that of acceleration is indeterminate.

3. Semiclassical analysis

The one-dimensional hydrogen atom poses an interesting problem for semiclassical consideration. A routine use of the Bohr–Sommerfeld quantization condition

$$\begin{aligned} 2\pi n\hbar &= \oint p\,\mathrm{d}x = 4\int_0^{x_m} p\,\mathrm{d}x = 4\int_0^{x_m} (2mE + 2me^2/x)^{1/2}\,\mathrm{d}x \\ &= 2\pi(2me^2x_m)^{1/2} = 2\pi(-2me^4/E)^{1/2} \end{aligned} \tag{8}$$

(where $x_m = -e^2/E > 0$ since energy is to be negative for bounded motions) yields the following energy levels:

$$E = -2me^4/\hbar^2 n^2 \tag{9}$$

which are four times those of the corresponding model in three dimensions and coincide with them only if $n \gg 1$ is restricted to take either even or odd values.

Although the simple calculation has been suggested by Lapidus (1988) as a supplementary example in an introductory quantum mechanics course, it is actually incorrect. The point is that WKB-approximation, on which (8) is based, does not apply in this case near the origin, because $\mathrm{d}(\hbar/p)/\mathrm{d}x = me^2\hbar(2mE + 2me^2/x)^{-3/2}/x^2 \sim x^{-1/2}$ is not small, but tends to infinity as $x \to 0$. Therefore one should be extremely careful with the semiclassical treatment of singular potential wells, since it might fail not only close to the turning points (where it normally does due to divergence of the particle's wavelength), but also for some points in between (due to divergence of the wavelength's derivative, in spite of the wavelength itself may vanish at the point). The familiar semiclassical solution in a potential well is defined by two complementary expressions:

$$\psi(x) = p^{-1/2}\begin{cases} C_1 \sin\left[(1/\hbar)\displaystyle\int_{-x_m}^{x} p\,\mathrm{d}x + \pi/4\right] & x > -x_m \\ C_2 \sin\left[(1/\hbar)\displaystyle\int_{x}^{x_m} p\,\mathrm{d}x + \pi/4\right] & x < x_m \end{cases} \tag{10}$$

where $p = (2me^2/|x| - 2m|E|)^{1/2}$. The condition (8) then stems from the requirement that both parts of equation (10) are to represent the same wavefunction, and therefore they must be equal at any point $-x_m < x < x_m$.

However, in the case under consideration, neither of the functions (10) satisfies the Schrödinger equation in the region about the singular point of the potential for the violation of the WKB-approximation there, that makes their immediate matching illicit. To bypass the difficulty, the matching of solutions can be carried on through an intermediate function, for which another approximate (but not semiclassical) solution in the region of singularity may be taken. For the same reason, a similar procedure is usually followed near the turning points for matching the respective semiclassical solutions inside and outside a potential well (close to its walls) to get the expressions (10). In the immediate vicinity of the origin, $|E| \ll e^2/|x|$, and the Schrödinger equation reduces to:

$$\psi'' + (\alpha/|x|)\psi = 0 \tag{11}$$

where $\alpha = 2me^2/\hbar^2$.

By the change of variables seen from equation (12) below it is transformable to the Bessel equation, cf 8.491(7) in Gradsteyn and Ryzhik (1965), whose normalizable odd solution,

$$\psi = \epsilon(x)\sqrt{|x|}J_1(2\sqrt{\alpha|x|}) \tag{12}$$

is seen to satisfy equation (11) by a direct substitution.

On the other hand, the even analogue of (12),

$$\psi_e = \sqrt{|x|}J_1\left(2\sqrt{\alpha|x|}\right) = \sqrt{x\epsilon(x)}J_1\left(2\sqrt{\alpha x\epsilon(x)}\right) \tag{13}$$

is incompatible with equation (11), since a formal differentiation yields:

$$\psi_e'' = 2\delta(x)\sqrt{\alpha}J_0\left(2\sqrt{\alpha|x|}\right) - \left(\alpha/\sqrt{|x|}\right)J_1\left(2\sqrt{\alpha|x|}\right).$$

The extra term here with $2\delta(x) = \epsilon'(x)$ does not vanish at the origin because $J_0(2\sqrt{\alpha|x|}) \to 1$ as $x \to 0$. Therefore, only odd solution (12) really satisfies the original equation (11).

The semiclassical functions (10) are to be compared (in respective regions) with the asymptotic form of equation (12) for large values of argument (very small $\hbar$),

$$\psi \approx C\epsilon(x)|x|^{1/4}\sin\left(2\sqrt{2me^2|x|}/\hbar - \pi/4\right) \qquad |x| \gg 1 \tag{14}$$

but not with each other.

As $|x| \to 0$, i.e. close to the origin, in expressions (10) the following approximations should be made:

$$\begin{aligned}\int_x^{x_m} p\,\mathrm{d}x &\approx \int_0^{x_m} p\,\mathrm{d}x - \int_0^x (2me^2/x)^{1/2}\,\mathrm{d}x \\ &= (1/2)\int_{-x_m}^{x_m} p\,\mathrm{d}x - 2\sqrt{2me^2x} \qquad x \ll 1.\end{aligned}$$

With this result one obtains the following expressions for the semiclassical solution near the singular point of the potential:

$$\psi(x) \approx |x|^{1/4}\begin{cases} C_1\sin\left(\dfrac{2}{\hbar}\sqrt{-2me^2x} - \dfrac{\pi me^2}{\hbar\sqrt{2m|E|}} - \dfrac{\pi}{4}\right) & x \leqslant 0 \\ C_2\sin\left(\dfrac{2}{\hbar}\sqrt{2me^2x} - \dfrac{\pi me^2}{\hbar\sqrt{2m|E|}} - \dfrac{\pi}{4}\right) & x \geqslant 0 \end{cases} \tag{15}$$

where the value derived in (8) has been used for $\oint p\,\mathrm{d}x$.

The equations (15) and (14) describe the same wavefunction only if

$$\frac{\pi me^2}{\hbar\sqrt{2m|E|}} = k\pi \qquad |E| = \frac{me^4}{2\hbar^2 k^2} \qquad C_1 = -C_2 = C(-1)^k. \tag{16}$$

The energy levels defined by (16) correspond to the familiar series for a three-dimensional Coulomb system and coincide with those given by (9) only for even $n = 2k$. This is in accord with the Fourier expansion (6) of the one-dimensional classical motion which contains only odd harmonics, whose frequency spacing turns out to be the same as for three-dimensional Kepler orbits in expansion (7).

Incidentally, if one would multiply equation (11) by x (or $|x|$), the even function (13) also becomes a solution of the 'tampered' equation, since the extra δ-function in ψ'' is found then ineffective (being now multiplied by x). The energy levels (16) remain intact, but the spectrum turns out to be doubly degenerate in conflict with the general result for one-dimensional motion (recall that the corresponding proof is independent of a potential's shape). Thus, such a premultiplication is to be considered as inadmissible in the case of a singular potential (cf section 5), while it is quite legitimate with non-generalized functions. The role of odd and even wavefunctions in the case under consideration is also discussed in the next section.

4. Matching of solutions in quantum mechanics

The Schrödinger equation of interest is

$$\psi'' - \left[\lambda - \frac{\alpha}{|x|}\right]\psi = 0 \tag{17}$$

where $\lambda = 2m|E|/\hbar^2, \alpha = 2me^2/\hbar^2$ and $E = -|E|$. Although, the wavefunction $\psi(x)$ should be defined in the whole range $|x| \leqslant \infty$, it is customary to treat the regions $x \geqslant 0$ and $x \leqslant 0$ separately and then match the respective solutions ψ_+ and ψ_- at $x = 0$. This is tantamount to writing the complete solution as

$$\psi(x) = \Theta(x)\psi_+(x) + \Theta(-x)\psi_-(x) \tag{18}$$

where $\Theta(x) = [1 + \epsilon(x)]/2$ is the unit step-function. It must be stressed that both ψ_+ and ψ_- are themselves defined over the whole range of x, the Θ-functions picking out their proper contributions to $\psi(x)$. In particular, even if $\psi_-(x)$ diverges as $x \to \infty$ and ψ_+ diverges as $x \to -\infty$ (which is often the case), the relation (18) still defines a normalizable function $\psi(x)$.

Differentiating equation (18) yields straightforwardly

$$\psi' = \Theta(x)\psi'_+ + \Theta(-x)\psi'_- + \delta(x)(\psi_+ - \psi_-) \tag{19a}$$

$$\psi'' = \Theta(x)\psi''_+ + \Theta(-x)\psi''_- + \delta(x)(\psi'_+ - \psi'_-). \tag{19b}$$

The relation $\delta'(x)f(x) = -\delta(x)f'(x)$ has been used above. Equation (17) now reads

$$\Theta(x)\left[\psi''_+ - \lambda\psi_+ + \frac{\alpha}{x}\psi_+\right] + \Theta(-x)\left[\psi''_- - \lambda\psi_- - \frac{\alpha}{x}\psi_-\right] + \delta(x)(\psi'_+ - \psi'_-) = 0. \tag{20}$$

Before continuing we should make the following observation concerning the role of generalized functions in any attempt at a piecewise solution of Schrödinger equation. The derivation of (20) illustrates that such a solution will in general enlist generalized functions, however, they can be handled easily because frequently one deals with potentials that are (a) non-singular, when both ψ and ψ' are continuous and the δ-function term drops out;

(b) symmetric, when $\psi(x)$ becomes an eigenfunction of parity and we can always choose $\psi_- = \pm\psi_+$, so that $\psi(x)$ can be constructed as

$$\psi(x) = \begin{cases} \psi_+(|x|) & \text{even } [\Theta(x) + \Theta(-x) = 1] \\ \epsilon(x)\psi_+(|x|) & \text{odd } [\Theta(x) - \Theta(-x) = \epsilon(x)] \end{cases} \tag{21}$$

where, now, ψ_+ must be considered for positive values of x only. Hence, one is required to solve only for ψ_+ in the range $x \geqslant 0$, which means that the Θ-functions in equations like (20) no longer have a role to play in the solution.

Returning to our problem, note that equation (21) still applies, even though the potential function is singular at $x = 0$. Integrating equation (17) over a small interval $(-\sigma, \sigma)$, $\sigma \to 0$, we find the condition

$$\psi'(0^+) - \psi'(0^-) = \alpha \int_{-\sigma}^{\sigma} \frac{\psi(x)}{|x|}\,\mathrm{d}x \tag{20$'$}$$

that must be obeyed. Since the integrand here may be singular, the result of integration is arbitrary, with the outcome depending on the prescription adopted to deal with the singularity. (Basically, conflicting claims in literature originate in different choices for the value of this integral.) We believe it is in this sense that the quantum mechanics of the problem in hand needs additional specification, although the approach based on integral transforms (to be discussed in the sequel) needs no such supplementation and hence may provide the more natural tool for investigating singular potential.

In the pursuit of a desirable prescription that would enable us to deal with the singularity, we turn to classical mechanics. We have already learnt in section 2 that if $-e^2/|x|$ is regarded as a potential energy of a physical system rather than merely 'a term' in the differential equation, then, it must correspond to a zero classical force at the equilibrium position $x = 0$. An inspection of equation (1) immediately suggests that this be ensured provided

$$\left\{\frac{1}{|x|}\right\}_{x=0} = 2\delta(x). \tag{22}$$

Using equation (22) and $f(x)\delta(x) = \delta(x)[f(0^+) + f(0^-)]/2$, equation (20$'$) translates into the relation

$$\psi'(0^+) - \psi'(0^-) = \alpha[\psi(0^+) + \psi(0^-)] \tag{23}$$

which provides an additional requirement permitting to pick out the correct solution from a host of competing formal solutions. Incidentally equation (22) automatically ensures the vanishing of the classical force at $x = 0$ in accord with the classical consideration of section 2.

Since the condition (23) at $x = 0$ has been obtained from the relation (22) between two generalized functions that is 'pointwise' and therefore not rigorous, it should be justified, say, by methods of self-adjoint operator extensions (see e.g., Reed and Simon, 1975). Generally there might be several self-adjoint extensions of the Hamiltonian corresponding to different 'physical' situations, and therefore the problem of a 'correct' choice of the self-adjoint extension is not just a question of mathematical 'technique' but it is closely related to the physics of the system under consideration (Reed and Simon, 1975, chapter XI). A self-adjoint extension of the Hamiltonian is defined by the 'boundary conditions' at $x = 0$, see, for example, equation (3.1.9) in the book (Albaverio *et al* 1988), and our relation (23) just corresponds to a particular choice of the constant in this condition (for continuous wavefunctions). Thus the method justifies an introduction of the 'zero-range' potential in our equation (22). Moreover, a consideration of deficiency indices of the operator in the same book by Albaverio *et al* (1988) shows that a self-adjoint extension with another type of zero-range potential including the derivative of delta-function is also possible in the one-dimensional case that actually justifies our manipulations with generalized functions in (19) and (20).

The one-dimensional 'Coulomb' problem has already been solved by Gostev *et al* (1987) using the results of self-adjoint operator extensions (Gostev *et al* 1986, 1987, Gostev and Frenkin 1988). In those papers some other 'induced' (or 'residual') zero-range potentials were considered, but instead of our equation (22) which is in accord with the classical motion in the same field (section 2), another condition of 'minimum number of delta-functions in the induced zero-range potential' had been used. As a result the authors, in particular, regard even wavefunctions as admissible solutions to the problem, but their approach is equivalent to a supplementation of the original potential $|x|^{-1}$ with a highly singular zero-range potential $\ln|x|\delta(x)$ and actually comes to another problem. On the other hand, our equation (22) means just a 'specification' of the original potential 'at the point of singularity' but not an addition of another term to it. Incidentally one more induced zero-range potential due to the singularity of $|x|^{-\nu}$, $\nu > 0$, has been also found by Ezawa *et al* (1975) using the method of functional integration independently of the boundary condition (23).

Now, if we consider odd wavefunctions whose derivatives are even, and if we assume that $\psi(0) = 0$ (as it should normally be for an odd function), equation (23) is trivially satisfied. On the other hand, the derivatives of even functions are odd, so that the left-hand side of equation (23) is, in general, different from zero and hence $\psi(0)$ must be non-zero. This requirement rules out the even functions of Loudon (1959) that contain the factor $|x|$ making $\psi(0) = 0$. With these functions removed from contention, the degeneracy problem disappears as well. Furthermore, if $\psi(0)$ is finite, $\psi'(0)$ according to equation (23) cannot be infinite. Hence, the even solutions of Heines and Roberts (1969) whose derivatives at the origin are infinite, are also rendered inadmissible. The fate of the odd analogue of their solutions that yield a continuous spectrum remains to be discussed.

To this end, we investigate the possible limiting behaviour of ψ near the origin. From equation (17) this is easily seen to be either

$$\psi_1 \sim x\left(1 - \frac{\alpha x}{2}\right)$$

or

$$\psi_2 \sim 1 - \alpha x \ln x .$$

The considerations above have already ruled out the possibility of even solutions. An odd solution with $\psi(x) \sim x(1 - \alpha|x|/2)$ near the origin, complies with the limiting requirement (equation (23)). We shall see that such functions correspond to a discrete negative energy spectrum. We now ask if solutions with a ψ_2-type limiting behaviour are able to generate odd solutions with a continuous spectrum. The only option one can rationally think of is a small distance behaviour $\epsilon(x)[1 - \alpha|x|\ln|x|]$, but then, as can be readily checked, ψ'' does not behave like $\psi/|x|$ near the origin as it should according to equation (17). Thus, neither a continuous spectrum nor an even solution of any kind is compatible with our requirement (22) leading to equation (23).

To complete a solution of equation (17), keeping in mind the large distance behaviour of ψ, we set

$$\psi = \phi(x)\exp\left[\frac{-\beta|x|}{2}\right] = \phi(x)\exp\left[\frac{-\beta\epsilon(x)x}{2}\right]. \tag{24}$$

The parameter β will be specified shortly, and $\phi(x)$, as yet, an unknown function, obeys

$$\phi'' - \beta\epsilon(x)\phi' + \left[\frac{\alpha}{|x|} - \beta\delta(x)\right]\phi = 0. \tag{25}$$

In writing equation (25) we have chosen to set $\beta^2/4 = \lambda$. Moreover, in view of the short distance behaviour of ψ (guaranteeing that it cannot diverge in the limit $|x| \to 0$), we have ignored a term containing $x\delta(x)$ in the expression for ψ'. It is seen from equation (25) that the additional zero-range potential $\beta\delta(x)$ appears at $x = 0$ again, but now due to the assumed dependence of the wavefunction on $|x|$ justified by its asymptotic behaviour as $x \to \pm\infty$. Such a term can be thought to belong to the original potential since the proper generalized function $|x|^{-1}$ actually 'contains' an arbitrary multiple of $\delta(x)$, see, for example, Lighthill (1975, section 3.3). An exact meaning to such a term can be given again by the methods of self-adjoint operator extensions.

It is interesting to notice incidentally that if we replace the original potential $-e^2/|x|$ by $-v_0\delta(x)$, $v_0 =$ constant, in equation (17), we would recover from equation (25) the only permitted solution to the zero-range potential problem of δ-functional well, with the correct energy level $E = -mv_0^2/2\hbar^2$ and the wavefunction corresponding to $\phi(x) =$ constant, see Albaverio *et al* (1988, theorem 3.1.4), while any other ϕ would give a non-normalizable solution. It will be readily appreciated that this is not the conventional manner in which the δ-potential problem is treated since no 'boundary condition' at $x = 0$ need to be explicitly used.

Recalling that an admissible $\phi(x)$ behaves as x near the origin so that the δ-function term in equation (25) is inconsequential, it is elementary to write down the solution we seek. For finite β, it is

$$\psi_n(x) = B_n x L_n^1(\beta_n|x|) \exp\left[\frac{-\beta_n|x|}{2}\right] \tag{26}$$

where, B_n are normalization constants, L_n^1 are associated Laguerre polynomials, and β_n are given by

$$\beta_n = \frac{\alpha}{n+1} \qquad n = 0, 1, 2, \ldots.$$

Hence, the allowed energy levels are given by

$$E_n = -\frac{me^4}{2\hbar^2(n+1)^2} \tag{27}$$

and form a non-degenerate discrete-energy spectrum the same as in equation (16).

5. The integral transform approach

An integral transform of the wavefunction is defined by its behaviour over the whole range of the argument. Hence there is no need to appeal to any matching of separate solutions for $x > 0$ and $x < 0$.

Employing a generalized Laplace type transform (Spain and Smith 1970) we try a solution to equation (17) of the form

$$\psi(x) = \int_C \exp[-f(x)t]u(t)\,\mathrm{d}t. \tag{28}$$

Here, the contour C of the integration (which is in the complex t-plane) as well as the functions, $f(x)$ and $u(t)$, are additional 'degrees of freedom' unspecified as yet. Given that the potential depends on $|x|$, a natural choice for $f(x)$ is $|x| = \epsilon(x)x$. Defined in this way, $\psi(x)$ is an even function of x, corresponding to the choice $\psi = \psi_+(|x|)$ in equation (21). Equation (28) then yields

$$\psi'' = \int_C [t^2 - 2t\delta(x)]\mathrm{e}^{-|x|t} u(t)\,\mathrm{d}t. \tag{29}$$

Substituting equations (28) and (29) in equation (17) and writing $\lambda = \beta^2/4$, we now have

$$\int_C \left[t^2 - \frac{\beta^2}{4} + \frac{\alpha}{|x|} - 2t\delta(x)\right] u(t)\mathrm{e}^{-|x|t}\,\mathrm{d}t = 0. \tag{30}$$

Here, again, a zero-range potential $2t\delta(x)$ emerges (whose intensity depends on the variable t) due to the same reason as in equation (25), whereas for an odd wavefunction the corresponding terms effectively vanish, see equations (32) and (33).

However, equation (30) for $x = 0$ is different from that for $x \neq 0$. It follows immediately that a function $u(t)$ that is independent of x, cannot satisfy it. In accord with our previous conclusions, we see once more that an even function of this type cannot satisfy equation (17). Left with the sole alternative $\psi = \epsilon(x)\psi_+$ in equation (21), we now try

$$\psi(x) = \epsilon(x) \int_C \mathrm{e}^{-|x|t} u(t)\,\mathrm{d}t \tag{31}$$

so that

$$\psi'' = \int_C [2\delta'(x) - 4t\epsilon(x)\delta(x) + \epsilon(x)t^2]\mathrm{e}^{-|x|t} u(t)\,\mathrm{d}t. \tag{32}$$

The product of two generalized functions $\epsilon(x)\delta(x)$ is undefined. Here we adopt the natural choice to set it equal to zero because such a choice is not contradicted by any known fact. For example, if we multiply $\epsilon(x)\delta(x)$ by a 'good' even function, $\exp[-|x|t]$, and integrate from $-\sigma$ to σ letting $\sigma \to 0$, then the integrand being odd and the integration being symmetric, the integral vanishes. (In other words, the principal value of the integral is zero.) Also, this choice is in harmony with the value $\Theta(0) = \frac{1}{2}$ recommended by Jauch and Rohrlich (1980), if one remembers that $\epsilon(x) = 2\Theta(x) - 1$. Finally, differentiating the relation $\epsilon^2(x) = 1$, the result $\epsilon(x)\delta(x) = 0$ emerges again.

Employing the relation $f(x)\delta'(x) = -\delta(x)f'(x)$, we now arrive at the equation

$$\int \left[t^2 - \frac{\beta^2}{4} + \frac{\alpha}{|x|}\right] u(t)\mathrm{e}^{-|x|t}\,\mathrm{d}t = 0. \tag{33}$$

Using the fact that $\exp[-|x|t]\,\mathrm{d}t$ as a function of t is equal to $-\mathrm{d}\,[(1/|x|)\exp(-|x|t)]$ and integrating the first term in equation (33) by parts, one now has

$$\frac{1}{|x|}\int_C \left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(t^2 - \frac{\beta^2}{4}\right)u(t)\right] + \alpha u(t)\right\}\mathrm{e}^{-|x|t}\,\mathrm{d}t = 0 \tag{34}$$

provided,

$$\frac{1}{|x|}\Delta_C\left[\mathrm{e}^{-|x|t}\left(t^2 - \frac{\beta^2}{4}\right)u(t)\right] = 0 \tag{35}$$

where Δ_C stands for the change in the value of the function in going from one end of the contour to the other.

To satisfy equation (34), one can take for $u(t)$ the general solution of the simple differential equation making the braces in equation (34) equal to zero:

$$u(t) = B\left(t+\frac{\beta}{2}\right)^{\alpha/\beta}\left(t-\frac{\beta}{2}\right)^{-\alpha/\beta}\left(t^2-\frac{\beta^2}{4}\right)^{-1} \qquad B = \text{constant} \tag{36}$$

whence,

$$\psi(x) = B\epsilon(x)\int_C \mathrm{e}^{-|x|t}\left(t+\frac{\beta}{2}\right)^{\nu-1}\left(t-\frac{\beta}{2}\right)^{-\nu-1}\mathrm{d}t \qquad \nu = \frac{\alpha}{\beta} > 0 \tag{37}$$

provided

$$\frac{1}{|x|}\Delta_C\left[\mathrm{e}^{-|x|t}\left(t+\frac{\beta}{2}\right)^{\nu}\left(t-\frac{\beta}{2}\right)^{-\nu}\right] = 0. \tag{38}$$

Finally, the contour C should be chosen to satisfy equation (38). If $x \neq 0$, there are two obvious choices. First, a contour that starts at infinity, goes once around the point $t = \beta/2$ and returns to infinity, so that the exponential takes the zero values at its ends ($\beta > 0$ for a normalizable function). Second, a contour that starts at infinity and terminates at the point $t = -\beta/2$ where the first bracket vanishes. The integrals appearing in equation (37) corresponding to the two contours mentioned can be expressed in terms of Whittaker functions, using their integral representations (Whittaker and Watson 1973). These possibilities result in no discretization of energy, so that odd solutions analogous to the even ones of Heines and Roberts (1969) (that lead to a continuous spectrum) are the outcome.

However, for $x = 0$, neither of these contours comply with the requirement imposed by equation (38), since its left-hand side manifestly diverges. The only way to save the situation for *all* x is to avoid going to infinity and choose, instead, a closed (finite) contour C surrounding the point $t = \beta/2$, restricting ν to be an integer. This makes $t = \beta/2$ a pole instead of being a branch point. Equation (38) is now trivially and identically satisfied, the end points of the contour C being the same and the function of interest turning single valued. Thus, $\beta = \alpha/\nu = \alpha/(n+1) = \beta_n$, $n = 0, 1, 2, \ldots$, reproducing the spectrum reported in equation (27). (The case $\nu = 0$, $\beta = \infty$, that has been excluded here, will be commented upon in due course.)

Using residue theory, the wavefunctions are straightforwardly found to be

$$\psi_n(x) = \frac{B_n\epsilon(x)}{(n+1)!}\mathrm{e}^{\beta_n|x|/2}\frac{\mathrm{d}^{n+1}}{\mathrm{d}\beta_n^{n+1}}(\beta_n^n\mathrm{e}^{-\beta_n|x|}). \tag{39}$$

Invoking Rodrigues representation for the Laguerre polynomials, equation (39) can be verified to be the same as equation (26), as should be the case. The result takes a more conventional form if the relation

$$\left(t+\frac{\beta}{2}\right)^{\nu}\left(t-\frac{\beta}{2}\right)^{-2-\nu}\mathrm{d}t = -[(\nu+1)\beta]^{-1}\,\mathrm{d}\left[\left(t+\frac{\beta}{2}\right)^{\nu+1}\left(t-\frac{\beta}{2}\right)^{-1-\nu}\right]$$

is used before integrating equation (37) by parts and the residue theorem applied subsequently.

The above demonstration accomplishes the principal goal of this work. However, with the aim of shedding more light on the problem we try to understand, how Davtyan *et al* (1987) employing a Fourier transform approach (momentum representation), happened to discover an additional set of levels with even parity and degenerate with the ones reported here. To this end, it is convenient to introduce the following self-evident representation for the potential in hand, namely:

$$\frac{1}{|x|} = \int_0^\infty \mathrm{e}^{-\gamma|x|}\,\mathrm{d}\gamma. \tag{40}$$

Let

$$\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}px} a(p)\,\mathrm{d}p. \tag{41}$$

The momentum function $a(p)$ is found to satisfy the integral equation

$$a(p) = \frac{\mathrm{i}\alpha}{2\pi}\frac{1}{p^2+\beta^2/4}\int_0^\infty \epsilon(\gamma)\,\mathrm{d}\gamma\int_{-\infty}^{\infty}\frac{a(w)\,\mathrm{d}w}{w-p+\mathrm{i}\gamma}. \tag{42}$$

The convolution theorem and equation (40) have been used in equation (17) to arrive at the above expression for $a(p)$. Assuming that $a(p)$ has no essential singularities (only poles) and closing the contour by an infinitely large semi-circle in the complex w-plane, we recover our previous solutions with the anticipated quantization condition $\beta_n = \alpha/(n+1)$. No degenerate spectrum materializes.

Now we focus on the difference between our procedure and that of Davtyan *et al* (1987). These authors premultiply equation (17) by $|x|$ prior to taking the Fourier transform which is a questionable operation when dealing with non-analytic functions (cf the last paragraph concluding section 3). Potentially, it is capable of inducing features that may be alien to the original equation. We illustrate this below. Consider, for example, the Fourier transform of the function $\psi_0(x)$ for $n = 0$ (see equation (26)). Its even and odd counterparts are

$$\Phi\{\psi_{\mathrm{odd}}\} = \Phi\{x\,\mathrm{e}^{\frac{-\beta|x|}{2}}\} = \frac{\mathrm{i}\beta p}{(p^2+\beta^2/4)^2} = \mathrm{Im}\frac{1}{(p-\mathrm{i}\beta/2)^2} \tag{43a}$$

and

$$\Phi\{\psi_{\mathrm{even}}\} = \Phi\left\{|x|\mathrm{e}^{\frac{-\beta|x|}{2}}\right\} = \frac{p^2-\beta^2/4}{(p^2+\beta^2/4)^2} = \mathrm{Re}\frac{1}{(p-\mathrm{i}\beta/2)^2}. \tag{43b}$$

Direct substitution shows that equation (43a) satisfies equation (42) whereas equation (43b) does not. On the other hand both the above functions satisfy the corresponding equation of Davtyan *et al* (1987) wherein a premultiplication by $|x|$ has already been effected. Hence, the degeneracy that obtains in their work has been inadvertently injected from outside. This can be also seen from equation (25) whose odd solutions (that are required to vanish at the origin) remain immune to the addition of a δ-function perturbation to the potential (regardless of the sign and strength of the added term). However, such an addition may change entirely the status of even solutions. In particular, a δ-function term, like the last term in the square brackets of equation (25), can be improperly washed out by the aforementioned premultiplication, so that both even and odd functions will satisfy the modified equation, introducing an artefact degeneracy.

Finally, we return to infinite energy state ($\beta \to \infty$) that emerges through Loudon's (1959) treatment and could supposedly represent the ground state of the 1H atom. According to Loudon, the normalized wavefunction of such a state is $\sqrt{\beta/2}\exp(-\beta|x|/2)$, $\beta \to \infty$, i.e. the probability density corresponds to a δ-function distribution. (Stated crudely, the wavefunction behaves like the square-root of a δ-function.) Although the Schrödinger equation is trivially satisfied by the function for any $x \neq 0$, it is straightforward to check that for $x = 0$ it is not. This alone suffices to rule out any further consideration. Alternatively, substituting the above function in equation (17) and using equation (40), one must have

$$\alpha\int_0^\infty \mathrm{e}^{-|x|(\gamma+\frac{\beta}{2})}\,\mathrm{d}\gamma - \beta\delta(x)\mathrm{e}^{\frac{-\beta|x|}{2}} = 0. \tag{44}$$

But equation (44) cannot be satisfied at $x = 0$ that is seen by integrating it from $x = -\sigma$ to $x = +\sigma$ and letting $\sigma \to 0$, $\beta \to \infty$. Such a failure is consistent with the result we established earlier that even functions are not admissible as solutions.

However, one could suggest that, physically, the object of interest is $|\psi|^2$. Hence, going beyond Loudon, let us construct the odd function $\psi(x) = \epsilon(x)\sqrt{\beta/2}\exp(-\beta|x|/2)$ corresponding to the second option in equation (21). This leaves $|\psi|^2$ unaffected, since, $\epsilon^2 = 1$. Moreover, equation (17) is now trivially satisfied for all x, including the point $x = 0$. Technically, this liquidates our previous objection to the feasibility of the even parity state. However, as $\beta \to \infty$, the wavefunction tends to zero everywhere in a way that the δ-function probability distribution still holds. Such a possibility, to the best of our knowledge, has never been noticed in quantum mechanics previously. The Fourier transform of this function, $\psi(x) = \epsilon(x)\sqrt{\beta/2}\exp(-\beta|x|/2)$, is

$$a(p) = -\frac{\mathrm{i}p\sqrt{\beta}}{2\pi(p^2 + \beta^2/4)}$$

and vanishes everywhere in the limit $\beta \to \infty$. No wonder, such a function trivially satisfies both our and Davtyan *et al*'s (1987) integral equation. Basically, in the limit $\beta \to \infty$, the function $a(p)$ becomes zero identically, i.e. the Fourier transform of such a 'generalized' function makes no sense. For completeness, we note that our (as well as Davtyan *et al* 1987) momentum space equations are also satisfied by the original Loudon's even counterpart of the above function with $a(p)$ vanishing everywhere (as $\beta \to \infty$) again.

In our view, such a state should have been discarded on physical grounds alone. Although the corresponding function appears in the semiclassical limit, it cannot belong to the family of solutions given by Eq. (26). Were it not the case, the entire set of remaining states with finite energies should be inaccessible and the very discussion of the problem becomes less than meaningful. An elegantly simple mathematical argument due to Andrews (1966) reaches the same conclusion. In spite of it all, it has served, nevertheless, to raise a deeper question of a more general nature. It is this. Is a δ-function probability distribution admissible in quantum mechanics? However, an attempt to discuss this problem would lead us far beyond the scope of this paper.

6. Summary

The salient features of this study may be summed up as follows.

(i) The Bohr–Sommerfeld quantization condition does not apply as such to the case of a singular potential well which requires a comprehensive semiclassical analysis.

(ii) Contrary to repeated claims in the literature, the discrete energy levels of the 1H atom are non-degenerate. Only odd-parity solutions are admissible. The δ-function terms in the second derivatives of the even functions are essential and may not be ignored.

(iii) The results based on limiting procedures applied to smoothed potentials are not reliable in the case of singular potentials. Stated simply, given the solutions to the problem of a Hamiltonian $H(g)$, those of the Hamiltonian $H(0)$ are not necessarily recovered by letting $g \to 0$. For the case of singular potentials, this circumstance is not unfamiliar (Gesztesy 1980, Chhajlany 1992). On the other hand, proper equations involving generalized functions have been able to lead us to the correct result in the present case.

(iv) Multiplication of all terms in a differential equation by a function of the independent variable admits the risk of distorting the properties of its solutions, in cases when the equation contains singular, non-analytic coefficients and/or has non-analytic solutions. In particular, in the present example, the even functions that are not solutions to the original equation manage to satisfy the modified equation. Such operations are best avoided. The result of tampering with equation (17) has been to ascribe an $O(2)$ symmetry to a problem that indeed has none.

(v) For potentials containing an essential singularity, some supplementary guidance may be needed so that the singularity can be dealt with meaningfully. In our case, we have appealed to classical mechanics to extract the appropriate specification. This contradicts no dictum of quantum mechanics and as a bonus, ensures that its limiting case is consistently accommodated. Whether this is a universal recipe is not for this study to spell out, attention here being confined to a specific Hamiltonian. That a correct element was injected is borne out by the independent integral transform approach that has the merit of demanding no supplementary specification, explicit matching of solutions being not the means of extracting the quantization for it.

(vi) There is no scope for a continuous bound-state spectrum. This results from our insistence that the Schrödinger equation be satisfied for *all* x, ($x = 0$ included).

(vii) An even-parity nodeless function has been shown to be inadmissible. On the other hand, an odd solution corresponding to infinite binding energy turns out to represent a trivial solution of the integral equation in that the associated momentum function vanishes everywhere.

(viii) In effect, the 1H atom possesses only a non-degenerate spectrum with a Rydberg progression of levels that replicates the spectrum of a three-dimensional Coulomb system. Applied properly, the well established machinery of quantum mechanics suffices to handle this problem that features an essential singularity in the permitted domain of motion of the particle.

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